

## Rock Around the Clock with Circle Theorems

A 12-pin circular geoboard can be used to introduce, explore and represent circle theorem geometry. Having  $360^\circ$  in a full circle has been beneficial in many ways, since this neat number has many factors. Another number with many connections to the circle is 12, since a clock has 12 equally spaced points on its circular face. So it seems natural that combining the clock face and the concept of angle could allow us to explore areas of the mathematics curriculum such as circle theorems.

There are as many as 12 circle theorems on the curriculum, though we may only touch on a few of these in depth with pupils. Dynamic geometry software can obviously be used to demonstrate them, but some method of exploring them using paper & pencil methods and a protractor will also be important. A high quality, mathematically beautiful record on paper emulating the dynamic content they have seen, which they can glue in their books, or stick to the fridge at home, has its own advantages.

One of the best things about using a 12-pin circular geoboard (12-PCG) is that all the angles produced will be exact multiples of  $15^\circ$ , making them easy for pupils to measure. It is rewarding and reassuring that not only are their angles close to within  $1^\circ$  of the true value, but, for many who are careful with their measurement, they are 'dead-on'. These activities encourage pupils to be more accurate with their measurement and construction so that they too can impress with these essential skills.

### Triangles

So where to begin? A suitable starting point might be to look at cyclic triangles (figure 1). One might wish to introduce a notation for constructions at this point, as the shape has been formed by drawing three lines connecting the equally spaced pins. Hence this triangle ABC can be referred to as 4-7-9-4, adopting a clock-based numbering system beginning and ending with the lowest number. In addition to measuring the angles and recording them on the templates, one might also wish to introduce correct angle notation in discussion; hence, the  $105^\circ$  angle C is also angle ACB or  $\angle ACB$ .

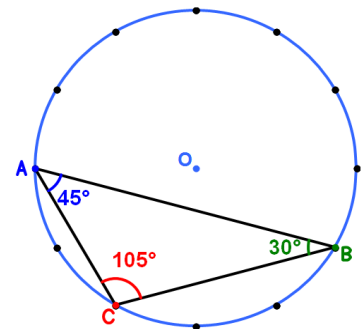


figure 1

Next, on a new 12-PCG, whilst keeping the line segment AB fixed, move vertex C from position 9 to 10, then 11, and record the new values of the angles each time. Figure 2 and figure 3 show the triangles created at each of these positions. Pupils may come up with the theorem themselves and, of course, we can explore further different cyclic triangles to confirm that the theorem "the angles subtended by a chord in the same segment are equal" applies in all cases.

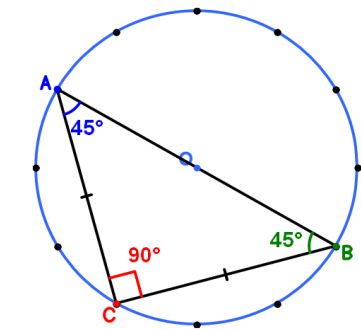


figure 2

One limitation of a 9-PCG is that you cannot construct a diameter through the centre of the circle. On a 12-PCG there are only 3 possible distinct unique triangles one can construct with a diameter as one of the sides though there are obviously congruent versions of these in different orientations.

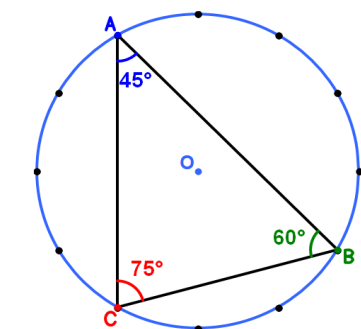


figure 3

The triangle shown in figure 4 has angles of  $30^\circ$ ,  $60^\circ$  and  $90^\circ$  and sides of length  $r$ ,  $\sqrt{3}r$  and  $2r$  opposite to these angles. Quite a lot can be made of this very simple diagram, which is easy to construct and measure from on a 12-PCG. You may wish to divide the triangle further. Drawing a line from O to B, produces an equilateral triangle on the right and an isosceles triangle on the left.

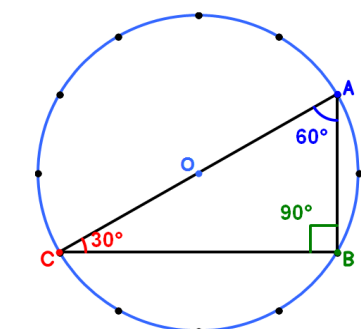


figure 4

Alternatively, you may wish to apply Pythagoras' Theorem or trigonometry to the original right-angled triangle to verify the value of all the angles and sides. Even the sine and cosine rule may rear their heads for other triangles.

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### Quadrilaterals

This previous right-angled triangle leads nicely to cyclic quadrilaterals. Ask pupils to visualise the effect of reflecting this right-angled triangle along its diagonal side BC. The shape produced by object and image is a kite (figure 5).

The Trinity Maths programme is intelligent enough to provide full geometric annotation of the shape dynamically. This powerful feature is something perhaps beyond the scope of any other form of dynamic geometry software available for use in the classroom.

The theorem "the sum of the opposite angles of a cyclic quadrilateral is  $180^\circ$ " is being demonstrated here. Of course there are naturally a large but limited number of possible cyclic quadrilateral permutations on a 12-PCG. One such possible permutation (figure 6) is a far more irregular shape than that shown in figure 5 yet it obeys the theorem.

Figure 7 is a demonstration of the theorem "the angle subtended at the centre is twice the angle at the circumference". Angle BOC ( $150^\circ$ ) is twice angle BAC ( $75^\circ$ ). Note that 3 of the 4 sides are equal in length. Perhaps an obvious next question would be: Is it possible to construct another different quadrilateral with one vertex at the centre also possessing 3 and only 3 equal sides?

Figure 8 shows such a quadrilateral with vertex A swapped over from the 8 o'clock position to 4 o'clock. Importantly, this demonstrates that in certain configurations a reflex angle is created. Again the angle subtended at the centre BOC ( $210^\circ$ ) is twice that at the circumference BAC ( $105^\circ$ ).

By moving vertex B clockwise 1 pin, we can construct a quadrilateral with one vertex at the centre, but this time it possesses 4 equal sides. Hence, we have a rhombus with 4 sides of length  $r$  (figure 9). Ask your pupils for another way of describing this shape. Certainly it is a parallelogram but it is also 2 equilateral triangles stuck back to back.

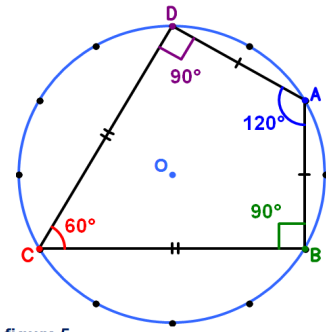


figure 5

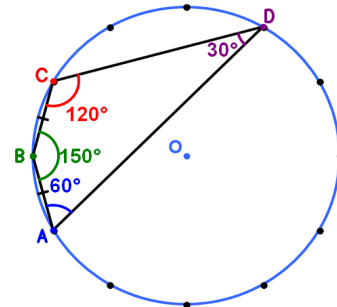


figure 6

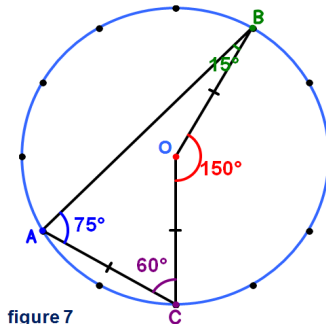


figure 7

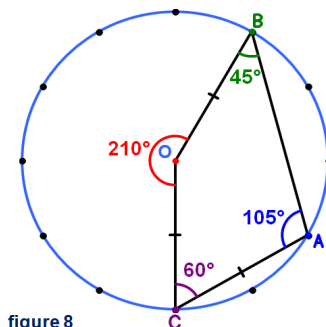


figure 8

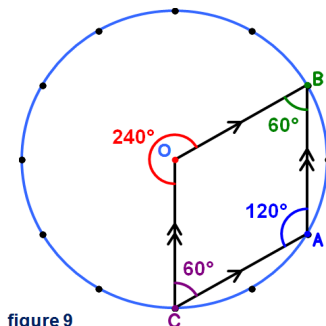


figure 9

## Rock Around the Clock with Circle Theorems

### Quadrilaterals (continued)

Moving vertices A and B one pin in the clockwise and anti-clockwise directions respectively produces a trapezium (figure 10). Obviously this shape has 2 pairs of allied (or interior) angles, although the diagram is showing the reflex conjugate of the obtuse angle BOC (which would be  $150^\circ$ ). I've participated in an illuminating discussion with some of my more able students and even abler teaching colleagues about whether there is a circle theorem waiting to be discovered which explains just why angles BAC ( $105^\circ$ ) and ACO ( $75^\circ$ ) are respectively  $90^\circ \pm \frac{1}{2}$  of angle ABO. The geometrical reasoning which follows can be quite involving. There are numerous definitions and assumptions which can be drawn out in a lesson which lead to a proof.

Before we leave quadrilaterals, it would be nice to devote some time to looking at the shapes formed when the lines overlap (figure 11). The geometrical reasoning behind these shapes will really test the class. It may be worthwhile and an even richer activity not to allow pupils to use in their explanations certain definitions, assumptions or some of the circle theorems they know or have found.

### Tangents and chords

The final theorem illustrated here is the one that says "the angle between a tangent and a chord through a point of contact is equal to the angle in the alternate segment". This theorem is in the higher GCSE tier and the literacy and vocabulary demands can present their own problems.

The diagram (figure 12) perfectly shows each of these aspects and these neat numerical values allow clear demonstration of the theorem. Moving vertex C anti-clockwise alters the shape's geometry. As angles CAT and ABC increase by  $15^\circ$ , BAC reduces by  $15^\circ$  (see figure 13 and figure 14). Annotation of this diagram with not only the measured angles but also the discrete words and phrases 'tangent', 'chord', 'point of contact' and 'alternate segment' has aided my pupils' understanding and application of this theorem to problem-solving questions in future lessons.

### Further work

Geoboards in their circular, rectangular and isometric forms are immensely useful for enabling accurate constructions and are undeniably rich for pupil exploration. The 12-PCG and its resemblance to a clock-face means that this is a very accessible activity for pupils of all levels, and able primary pupils could work on these tasks too.

As described above, the angles produced will be exact multiples of  $15^\circ$ . If a 9, 18 or 36-PCG were used instead of a 12-PCG, the angles produced would be exact multiples of  $20^\circ$ ,  $10^\circ$  or  $5^\circ$  respectively. The relationships between the number of pins and the angles produced can be explored and generalized for an n-pin circular geoboard. There is scope for even greater geometric, algebraic and trigonometric reasoning to be developed and applied.

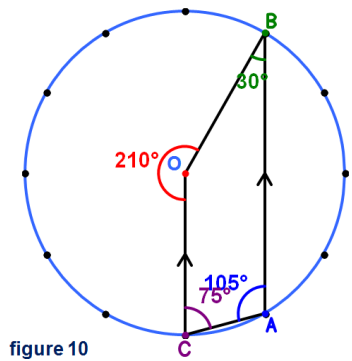


figure 10

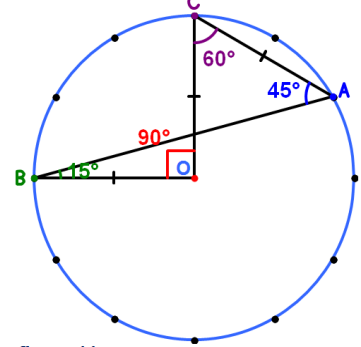


figure 11

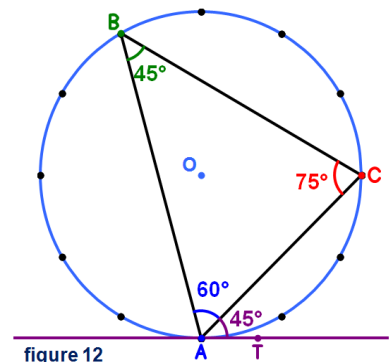


figure 12

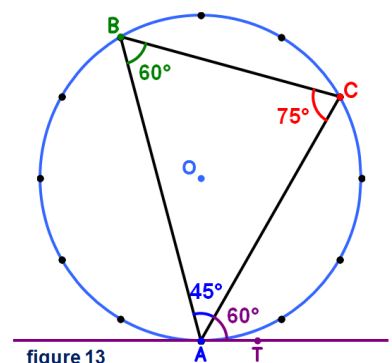


figure 13

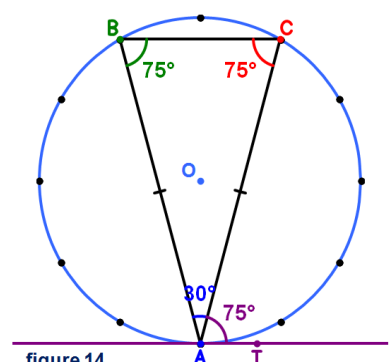


figure 14